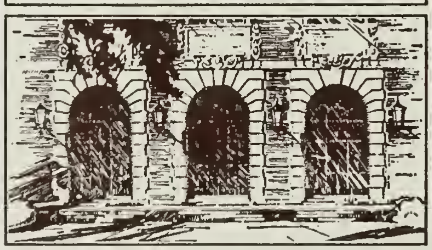

UNIVERSITY OF ILLINOIS
GRADUATE COLLEGE
DIGITAL COMPUTER LABORATORY

REPORT NO. 95

NUMERICAL QUADRATURE OVER A RECTANGULAR DOMAIN IN TWO OR MORE DIMENSIONS

by

J. C. P. Miller

December 11, 1959

(This is the third of a series of papers, all of which will be published in Mathematics of Computation)

This work was supported in part by the Office of Naval Research under Contract Nonr-1834(27).

Numerical Quadrature over a Rectangular Domain in Two or More Dimensions

By J.C.P. Miller

III Quadrature of a Harmonic Integrand

Introductory

1. In Note I(1), §5, formula (B), §7, formula (B'), and in §9, also in Note II(1) in several places, we have seen how the error term is very much reduced if the integrand $f(x,y)$ is a harmonic function, that is, if $\nabla^2 f = 0$. In this note we pursue further this special case, in which especially high accuracy is attainable with few points.

It may not be often that the integrand will have this special form, but it seems worthwhile to develop a few of the interesting formulae. We start by obtaining expansions for n variables, and more extensive ones for two variables, and then obtain and consider special quadrature formulae.

2. Expansions. As in II(1), §2, we develop $f(x_1, x_2, \ldots, x_n)$ as a Taylor series in even powers of each of the variables x_r. Then, using $\nabla^2 f_0 = 0$ whenever it is applicable, we obtain

$$
(2.1)\quad \left\{
\begin{aligned}
J = I/(2h)^n &= (2h)^{-n}\left(\int_{-h}^{h}\right)^n f(x_1,x_2,\ldots,x_n)\, dx_1,\, dx_2\ldots dx_n \\
&= f_0 + \frac{h^4}{5!}\,\frac{4}{3}\mathcal{D}^4 f_0 + \frac{h^6}{7!}\,\frac{16}{3}\mathcal{T}^6 f_0 + \frac{h^8}{9!}\left(\frac{16}{5}\mathcal{D}^8 + \frac{192}{5}\mathcal{Q}^8\right)f_0 \\
&\quad + \frac{h^{10}}{11!}\left(\frac{128}{3}\mathcal{D}^4\mathcal{T}^6 + \frac{1280}{3}\mathcal{P}^{10}\right)f_0 \\
&\quad + \frac{h^{12}}{13!}\left(\frac{64}{7}\mathcal{D}^{12} + \frac{4096}{35}\mathcal{T}^{12} + \frac{61184}{105}\mathcal{D}^4\mathcal{Q}^8 + \frac{707584}{105}\mathcal{S}^{12}\right)f_0 \\
&\quad + \ldots
\end{aligned}
\right.
$$

where, as before, extended,

$$
(2.2)\quad \mathcal{D}^4 f_0 = \Sigma\,\frac{\partial^4 f_0}{\partial x_r^2\,\partial x_s^2} \qquad \mathcal{T}^6 f_0 = \Sigma\,\frac{\partial^6 f_0}{\partial x_r^2\,\partial x_s^2\,\partial x_t^2} \qquad \mathcal{Q}^8 f_0 = \Sigma\,\frac{\partial^8 f_0}{\partial x_r^2\,\partial x_s^2\,\partial x_t^2\,\partial x_u^2}
$$

etc., the summations extending over all possible combinations of r, s, t,...with no two equal.

(1) J.C.P.Miller, M.T.A.C., in the press.

We have likewise, labelling the symmetrical sets of points as in Note II, the expansions for sums of values of f over the sets

(2.31) 0 $\qquad f_0$

(2.32) $\alpha(a)$
$$2nf_0 - \frac{4h^4a^4}{4!}\varpi^4 f_0 + \frac{6h^6a^6}{6!}\mathcal{J}^6 f_0 + \frac{4h^8a^8}{8!}(\varpi^8 - 2\mathcal{Q}^8) f_0$$
$$-\frac{10h^{10}a^{10}}{10!}(\varpi^4\mathcal{J}^6 - \rho^{10}) f_0 - \frac{2h^{12}a^{12}}{12!}(2\varpi^{12} - 3\mathcal{J}^{12} - 6\varpi^4\mathcal{Q}^8 + \mathcal{S}^{12}) f_0 + \ldots$$

(2.33) β (b)
$$2n(n-1)f_0 - \frac{8h^4b^4}{4!}(n-4)\varpi^4 f_0 + \frac{12h^6b^6}{6!}(n-16)\mathcal{J}^6 f_0$$
$$+\frac{8h^8b^8}{8!}\left\{(n+6)\varpi^8 - 2(n-64)\mathcal{Q}^8\right\} f_0 - \frac{20h^{10}b^{10}}{10!}\left\{(n-4)\varpi^4\mathcal{J}^6 - (n-256)\rho^{10}\right\} f_0$$
$$-\frac{4h^{12}b^{12}}{12!}\left\{2(n-34)\varpi^{12} - 3(n+362)\mathcal{J}^{12} - 6(n-728)\varpi^4\mathcal{Q}^8 + 6(n-1024)\mathcal{S}^{12}\right\} f_0 + \ldots$$

(2.34) γ (c,d)
$$4n(n-1)f_0 - \frac{8h^4}{4!}\left\{(n-1)(c^4+d^4) - 6c^2d^2\right\}\varpi^4 f_0 + \frac{12h^6}{6!}\left\{(n-1)(c^6+d^6) - 15c^2d^2(c^2+d^2)\right\}\mathcal{J}^6 f_0$$
$$+\frac{8h^8}{8!}\Big[\left\{(n-1)(c^8+d^8) - 28c^2d^2(c^4+d^4) + 70c^4d^4\right\}\varpi^8$$
$$-2\left\{(n-1)(c^8+d^8) - 28c^2d^2(c^4+d^4) - 70c^4d^4\right\}\mathcal{Q}^8\Big] f_0$$
$$+ \ldots$$

(2.35) $\mathcal{E}$ (e)
$$\frac{4}{3}n(n-1)(n-2)f_0 - \frac{8h^4e^4}{4!}(n-2)(n-7)\varpi^4 f_0 + \frac{12h^6e^6}{6!}(n^2-33n+122)\mathcal{J}^6 f_0$$
$$+\frac{8h^8e^8}{8!}\left\{(n-2)(n+13)\varpi^8 - 2(n^2-129n+1094)\mathcal{Q}^8\right\} f_0 + \ldots$$

We recall that 0 is the origin, or centre of the square, α (a) includes all points with one coordinate $\pm$ ah and the rest zero, β(b) has two coordinates each independently $\pm$ bh and the rest zero, γ (c,d) has one coordinate $\pm$ ch, another $\pm$ dh and the rest zero and finally $\mathcal{E}$ (e) has three coordinates each independently $\pm$ eh with the rest zero.

3. Expansions over a square are simpler since $\mathcal{J}^6 f_0$, $\mathcal{Q}^8 f_0$ etc. vanish. They can be obtained by analysis with the detached operators - in particular ϖ; we proceed to obtain expansions with general terms.

If $F(z) = u + iv$ is a function of a complex variable $z = x + iy$ then both u and v are harmonic functions satisfying $D_x^2\phi + D_y^2\phi = 0$. Likewise, if u is a harmonic function, it can be shown that v exists such that u + iv is a function of a complex variable. We then have

$$D_y F = iF' = iD_n F$$

and

(3.1) $$D_x\, D_y = \mathcal{D}^2 = iD_x^2 = -i\, D_y^2$$

In order to develop expansions we therefore substitute

(3.2) $$D_x = i^{-1/2}\mathcal{D} \quad D_y = i^{1/2}\mathcal{D}$$

Consider, firstly

(3.3) $$J = (2h)^{-2}\int_{-h}^{h}\int_{-h}^{h} f(x,y)dx\, dy = \frac{1}{4h^2\, D_x\, D_y}(e^{hD_x}-e^{-hD_x})(e^{hD_y}-e^{-hD_y})f_0$$

The operator is

(3.4) $$\begin{cases} \dfrac{\sinh hD_x \sinh hD_y}{h^2\, D_x\, D_y} = \dfrac{\sinh i^{-1/2}h\mathcal{D} \sinh i^{1/2}\, h\mathcal{D}}{h^2\,\mathcal{D}^2} \\ \qquad = \dfrac{1}{2}\;\dfrac{\cosh (i^{1/2}+ i^{-1/2})h\mathcal{D} - \cosh (i^{1/2}-i^{-1/2})h\mathcal{D}}{h^2\mathcal{D}^2} \\ \qquad = \dfrac{1}{2}\;\dfrac{\cosh \sqrt{2}\, h \quad -\cos \sqrt{2}\, h\mathcal{D}}{h^2\,\mathcal{D}^2} \end{cases}$$

whence

(3.5) $$J = \left(\frac{2}{2!} + \frac{2^3h^4}{6!}\mathcal{D}^4 + \frac{2^5h^8}{10!}\mathcal{D}^8 + \dots + \frac{2^{2r+1}h^{4r}}{(4r+2)!}\mathcal{D}^{4r} + \dots\right) f_0$$

(3.6) Likewise $$\begin{cases} \sum\limits_{\alpha(a)} f(x_\alpha, y_\alpha) = (e^{ahD_x} + e^{-ahD_x} + e^{ahD_y} + e^{-ahD_y})\, f_0 \\ \qquad = 2\,(\cosh ah\, D_x + \cosh ah\, D_y)\, f_0 \\ \qquad = 2\,(\cosh i^{-1/2}\, ah\mathcal{D} + \cosh i^{1/2}\, ah\mathcal{D})\, f_0 \\ \qquad = 4\, \cosh \frac{ah}{\sqrt{2}}\mathcal{D} \cos \frac{ah}{\sqrt{2}}\mathcal{D}\, f_0 \\ \qquad = 4\, [1 - \frac{a^4h^4}{4!}\mathcal{D}^4 + \frac{a^8h^8}{8!}\mathcal{D}^8 + \dots + (-1)^r \frac{a^{4r}h^{4r}}{(4r)!}\mathcal{D}^{4r} + \dots] f_0 \end{cases}$$

and

$$
(3.7)\quad
\begin{cases}
\sum\limits_{\beta(b)} f(x_\beta, y_\beta) = 4 \cosh bh\, D_x \cosh bh\, D_y\, f_0 \\[2ex]
\qquad = 4 \cosh i^{-1/2} bh \mathcal{D} \cosh i^{1/2}\, bh\, \mathcal{D}\, f_0 \\[2ex]
\qquad = 2\, (\cosh bh \sqrt{2}\, \varpi + u \quad bh \sqrt{2}\varpi)\, f_0 \\[2ex]
\qquad = 4\, [1 + \dfrac{2^2 b^4 h^4}{4!} \varpi^4 + \dfrac{2^4 b^8 h^8}{8!} \varpi^8 + \ldots + \dfrac{2^{2r} b^{4r} h^{4r}}{(4r)!} \varpi^{4r} + \ldots]\, f_0
\end{cases}
$$

We shall not use all the expansions given above in the present note, but it seems useful to set out the collected results for future use.

4. Lattice-point formulae over a Square. We consider first formulae in two variables, and start with 9 points, putting $a = b = 1$ and using the sets 0, $\alpha(1)$, $\beta(1)$. We write

$$(4.1)\quad J = I/4h^2 = A_0\, f(0,0) + \sum A_\alpha\, f\,(x_\alpha, y_\alpha) + \sum A_\beta\, f\,(x_\beta, y_\beta)$$

using (x_α, y_α) etc. as typical sets of coordinates.

Using (3.5) - (3.7), we equal coefficients of $\varpi^{4r} f_0$, $r = 0(1)2$.

This gives

$$
(4.2)\quad
\begin{cases}
A_0 + 4\, A_\alpha + 4\, A_\beta = 1 \\[1ex]
\quad - 4\, A_\alpha + 16\, A_\beta = \dfrac{4}{15} \\[1ex]
\quad + 4\, A_\alpha + 64\, A_\beta = \dfrac{16}{45}
\end{cases}
$$

with correction term $C = -\,(-4\, A_\alpha + 256\, A_\beta - \frac{64}{91})\ \frac{h^{12}}{12!}\, \varpi^{12}\, f_0$

We obtain the formula

$$
(4.3)\quad
\begin{array}{|ccc|}
\hline
7 & -32 & 7 \\
-32 & 1000 & -32 \\
7 & -32 & 7 \\
\hline
\end{array}
\ \div 900 \qquad \text{with main correction term} -\frac{1952}{1365}\,\frac{h^{12}}{12!}\,\varpi^{12}\, f_0
$$

This formula is remarkably good. With the example of Part I, we have, writing $J' = h^2 J$

$$J' = \frac{1}{4} \int_0^{1.2} \int_0^{1.2} \sin x \sinh y\, dx\, dy = \frac{1}{4}\,(1 - \cos 1.2)(\cosh 1.2 - 1)$$

$$\doteqdot 0.12922\ 70590\ 73675\ 11602$$

Formula (4.3) gives

$$J' \doteqdot 0.12922\ 70590\ 72834\ 11029$$

with $E = -0.0^{12}841\ 00573$ and $C = +0.0^{12}841\ 01633$.

5. <u>Five-point formulae</u>. The high precision of (4.3) suggests that formulae of lessor precision, with fewer points, may be useful. We use the first two of (4.2) and take one of A_0, A_α, A_β to be zero (i) $A_0 = 0$ gives an eight-point formula with relatively poor precision.

$$(5.1) \qquad \boxed{\begin{array}{ccc} 19 & 56 & 19 \\ 56 & 0 & 56 \\ 19 & 56 & 19 \end{array}} \quad \div 300 \qquad \text{with main correction term } -40 \frac{h^8}{9!} \mathfrak{D}^8 f_0$$

(ii) $A_\alpha = 0$ gives

$$(5.2) \qquad \boxed{\begin{array}{ccc} 1 & 0 & 1 \\ 0 & 56 & 0 \\ 1 & 0 & 1 \end{array}} \quad \div 60 \qquad \text{with main correction term } -\frac{32}{5} \frac{h^8}{9!} \mathfrak{D}^8 f_0$$

(iii) $A_\beta = 0$ gives

$$(5.3) \qquad \boxed{\begin{array}{ccc} 0 & -1 & 0 \\ -1 & 19 & -1 \\ 0 & -1 & 0 \end{array}} \quad \div 15 \qquad \text{with main correction term } +\frac{28}{5} \frac{h^8}{9!} \mathfrak{D}^8 f_0$$

We observe that (5.2) and (5.3) combine in the proportions $\frac{7}{15} : \frac{8}{15}$ to give (4.3), though without an error estimate! Likewise $\frac{7}{3} \times (5.2) - \frac{4}{3} \times (5.3)$ gives (B) of Note I, and an estimate for the correction, namely $-\frac{112}{5} \frac{h^8}{9!} \mathfrak{D}^8 f_0$ when (x,y) is harmonic.

Another combination, that of (5.2) and (5.3) in equal proportions gives a small correction term:

$$(5.4) \qquad \boxed{\begin{array}{ccc} 1 & -4 & 1 \\ -4 & 132 & -4 \\ 1 & -4 & 1 \end{array}} \quad \div 120 \qquad \text{with main correction term } -\frac{4}{5} \frac{h^8}{9!} \mathfrak{D}^8 f_0$$

Again $4 \times (5.2) - 3 \times (5.3)$ gives small multipliers:

$$(5.5) \qquad \boxed{\begin{array}{ccc} 1 & 3 & 1 \\ 3 & -1 & 3 \\ 1 & 3 & 1 \end{array}} \quad \div 15 \qquad \text{with main correction term } -\frac{212}{5} \frac{h^8}{9!} \mathfrak{D}^8 f_0$$

Evidently (4.3) is most precise, but simultaneous use of (5.2) and (5.3) gives an idea of the precision attained and readily yield the better result if desired. Formula (5.5) might be helpful with desk computing, but (5.1) has little to recommend it.

Numerical results for some of the formulae using the example of § 4 are as follows:

Formula	Result J'	$10^{10} \times E$	$10^{10} \times C$
(5.1)	0.12922 72986	+2395	-2396
(5.2)	0.12922 70974	+ 383	- 383
(5.3)	0.12922 70255	- 336	+ 335
I (B)	0.12922 71932	+1341	+1342
(5.4)	0.12922 70615	+ 24	- 24

6. <u>General n; $2n^2 + 1$ points.</u> We consider now the n-dimensional case, $n \geq 3$, using lattice points 0, $\alpha(1)$, $\beta(1)$. In this case the term in $\mathcal{J}^6 f_0$ is relevant, and the $\mathcal{Q}^8 f_0$ term will appear in the error, except when n = 3.

We equate coefficients of f_0, $\mathfrak{B}^4 f_0$, $\mathcal{J}^6 f_0$ in the expansions resulting from use of (2.1), (2.31)-(2.33) in (4.1). We obtain

$$(6.1) \qquad \begin{cases} A_0 + 2n\, A_\alpha + 2n(n-1)\, A_\beta = 1 \\ \quad - 4\, A_\alpha - 8(n-4)\, A_\beta = \frac{4}{15} \\ \quad + 6\, A_\alpha + 12(n-16)\, A_\beta = \frac{16}{21} \end{cases}$$

while $C = -\left\{4\, A_\alpha + 8(n+6)A_\beta - \frac{16}{45}\right\}\frac{h^8}{8!}\mathfrak{B}^8 f_0 + \left\{8A_\alpha + 16(n-64)A_\beta + \frac{192}{45}\right\}\frac{h^8}{8!}\mathcal{Q}^8 f_0$

These yield

$$(6.2) \qquad A_0 = \frac{-61n^2 + 931n + 3780}{3780} \qquad A_\alpha = \frac{61n - 496}{3780} \qquad A_\beta = \frac{-61}{7560}$$

with

$$C = \frac{1198}{945}\ \frac{h^8}{8!}\mathfrak{B}^8 f_0 + \frac{3619}{315}\ \frac{h^8}{8!}\mathcal{Q}^8 f_0$$

In particular

$$(6.33) \qquad n = 3 \qquad A_0 = \frac{12048}{7546} \qquad A_\alpha = -\frac{626}{7560} \qquad A_\beta = -\frac{61}{7560}$$

$$(6.34) \qquad n = 4 \qquad A_0 = \frac{13056}{7560} \qquad A_\alpha = -\frac{504}{7560} \qquad A_\beta = -\frac{61}{7560}$$

(6.35) $n = 5$ $A_0 = \frac{13820}{7560}$ $A_\alpha = -\frac{382}{7560}$ $A_\beta = -\frac{61}{7560}$

(6.36) $n = 6$ $A_0 = \frac{14340}{7560}$ $A_\alpha = -\frac{260}{7560}$ $A_\beta = -\frac{61}{7560}$

As a numerical illustration for $n = 3$ we consider

$$J = \frac{1}{8} I = \frac{1}{8} \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} \cos \frac{3}{4} x \cos y \cosh \frac{5}{4} z \, dx \, dy \, dz$$

$$= \frac{16}{15} \sin \frac{3}{4} \sin 1 \sinh \frac{5}{4} = 0.9800827$$

Formula (6.33) gives $J \doteq 0.9799734$ with $E = -0.0000109$ and $C = +0.0000110$.

This result is less spectacular than that of §4, for these reasons:

i) In §4, $h = 0.6$, here $h = 1$, and the correction term in (4.3) contains a high power of h.

ii) The correction term in (6.2) is of order h^8, that in (4.3) is of order h^{12}

iii) The higher the number of dimensions, the more individual terms there are in $\mathcal{D}^8 f$, $\mathcal{D}^{12} f$, etc. In (4.3) there is only one term in $\mathcal{D}^{12} f$, in (6.33) there are 9 in $\mathcal{D}^8 f$.

iv) The effect of larger interval h is enhanced by the use of the factor $\frac{5}{4}$, which exceeds unity, in $\cosh \frac{5}{4} Z$; this is only partially balanced by the factor $\cos \frac{3}{4} x$.

In spite of these points, the formula (6.2) seems a good one.

7. Quadrature over a square; specially chosen points. Since the expansions of §3 involve only cross-differences $\mathcal{D}^4 f_0$, it appears likely that use of sets of diagonal points β will be more profitable than attempts to use sets α. It turns out that sets $0, \alpha(a), \beta(b)$ and $0, \alpha(a)$ both fail to give real values of α if maximum precision is sought. On the other hand, we can get several formulae making use of any number of sets $\beta(b_p)$, $p = 0(1)\, r$, both with and without the point 0.

We start first with r sets $\beta\,(b_p)$, without the point 0. We have to find the $2r$ constants A_{β_p}, b_p satisfying the equations

(7.1) $$\sum_{p=1}^{r} 4A_{\beta_p} \, b_p^{4(s-1)} = \frac{1}{(2s-1)(4s-3)} = C_{s-1}, \quad s = 1(1)\, 2r$$

obtained by substitution of (3.5) — (3.7) in

(7.2) $$J = \sum A_{\beta_p} f(\pm b_p h, \pm b_p h)$$

and equating the coefficients of the first 2r coefficients of $\mathfrak{D}^4$. Sundry powers of 4 have been cancelled.

By familiar arguments, the b_p^4 are roots of the equation

$$(7.3) \qquad \begin{vmatrix} 1 & x & x^2 & x^r \\ c_0 & c_1 & c_2 & c_r \\ c_1 & c_2 & c_3 & c_{r+1} \\ & & & \\ c_r & c_{r-1} & c_{r-2} & c_{2r-1} \end{vmatrix} = 0$$

These are the orthogonal polynomials corresponding to the weight function $w(x) = \frac{1}{2}(x^{-3/4} - x^{-1/2})$ for range $0 \leq x \leq 1$. The first two are

$$(7.4) \qquad \begin{cases} 15x - 1 = 0 \\ 819x^2 - 438x + 11 = 0 \end{cases}$$

The main correction term is obtained from the next power of $\mathfrak{D}^4$ and yields

$$(7.5) \qquad C = (c_{2r} - \sum_{p=1}^{r} 4\, A_{\beta_p}\, b_p^{8r}) \; \frac{2^{4r} h^{8r} \mathfrak{D}^{8r}}{(8r)!}$$

If the point 0 is included, our equations (7.1) are replaced by

$$(7.6) \qquad \begin{cases} A_0 + 4\sum_{p=1}^{r} A_{\beta_p} = 1 \\ \\ \sum_{p=1}^{r} 4\, A_{\beta_p}\, b_p^{4s} = \frac{1}{(2s+1)(4s+1)} = c_s, \quad s = 1(1)\, 2r \end{cases}$$

and the b_p^4 are roots of the equation

$$\begin{vmatrix} 1 & x & x^2 & x^r \\ c_1 & c_2 & c_3 & c_{r+1} \\ c_2 & c_3 & c_4 & c_{r+2} \\ & & & \\ c_{r+1} & c_{r+2} & c_{r+3} & c_{2r} \end{vmatrix} = 0$$

which are the orthogonal polynomials for the weight function $w(x) = \frac{1}{2}(x^{1/4} - x^{1/2})$ for the range $0 \leq x \leq 1$. The first two are

$$
\begin{aligned}
&3x - 1 = 0 \\
(7.8) \qquad & \\
&17017x^2 - 13650x + 1745 = 0
\end{aligned}
$$

The main correction term is this time

$$(7.9) \qquad C = \left(C_{2r+1} - \sum_{p=1}^{r} 4\, A_{\beta_p}\, b_p^{8r+4}\right) \frac{2^{2r+1}\, h^{4r+2}}{(4r+2)!} \mathcal{D}^{4r+2}$$

In each case the coefficients A_{β_r} may be computed by standard methods.

8. Formulae for r = 1. These have 4 and 5 points respectively

$$(8.1) \quad A_\beta = \frac{1}{4} \qquad b = 15^{-1/4} \qquad C = \frac{64}{225} \; \frac{h^8 \mathcal{D}^8}{8!} f_0$$

$$(8.2) \quad A_0 = \frac{4}{5} \qquad A_\beta = \frac{1}{20} \qquad b = 3^{-1/4} \qquad C = \frac{2816}{12285} \; \frac{h^{12} \mathcal{D}^{12}}{12!} f_0$$

Written out in full:

$$(8.3) \quad J = \frac{1}{4} I = \frac{1}{4}\left\{f(15^{-1/4}, 15^{-1/4}) + f(-15^{-1/4}, 15^{-1/4}) + f(15^{-1/4}, -15^{-1/4}) + f(-15^{-1/4}, -15^{-1/4})\right\}$$

$$(8.4) \quad J = \frac{1}{4} I = \frac{4}{5} f(o,o) + \frac{1}{20}\left\{f(3^{-1/4}, 3^{-1/4}) + f(-3^{-1/4}, 3^{-1/4}) + f(3^{-1/4}, -3^{-1/4}) + f(-3^{-1/4}, -3^{1/4})\right\}$$

As a numerical test use

$$J = \frac{1}{4} I = \frac{1}{4} \int_{-1}^{1} \int_{-1}^{1} \cos x \cosh y \, dy = \sin 1 \sinh 1$$

$$= 0.98889\ 77057\ 62865$$

Formula (8.3) gives 0.98889 06525
with $E = 0.00000\ 70533$ and $C = +0.00000\ 70547$
and formula (8.4) gives 0.98889 77062 41358
with $E = +0.0^9 4\ 78493$ and $C = -0.0^9 478543$.

9. Formulae for r = 2. These have 8 and 9 points respectively

$$
\begin{aligned}
(9.1) \quad b_1 &= 0.40316\ 26030\ 59346\ 89754 & A_{\beta_1} &= 0.22912\ 30654\ 28169\ 97222 \\
b_2 &= 0.84439\ 75319\ 23478\ 74713 & A_{\beta_2} &= 0.02087\ 69345\ 71830\ 02778
\end{aligned}
$$

with main correction term $\frac{54592}{57014685} \times \frac{256}{16!} h^{16} \mathcal{D}^{16} f_0$

(9.2) $b_0 = 0$ $\qquad A_0 = 0.69521\ 80834\ 12925\ 81989$

$b_1 = 0.63205\ 02078\ 18796\ 99524$ $\qquad A_{\beta_1} = 0.06686\ 42185\ 46105\ 38162$

$b_2 = 0.89531\ 63791\ 24106\ 97730$ $\qquad A_{\beta_2} = 0.00993\ 12606\ 00663\ 16340$

with main correction term $\frac{16832}{78975897} \times \frac{1024}{20!}\ h^{20}\ \mathfrak{D}^{20}\ f_0$

For $J = \frac{1}{4} \int_{-1}^{1} \int_{-1}^{1} \cos x \cosh y \, dx \, dy$

formula (9.1) gives 0.98889 77057 62853 38396

with $E = -0.0^{13}\,1171243$ and $C = +0.0^{13}1171555$

while formula (9.2) gives 0.98889 77057 62865 09647

with $E = +0.0^{19}9$ and $C = 0.0^{19}90$

With formula (9.2) we find approximately

0.82447 37090 77903 16756 for

$$\frac{1}{16} \int_{-2}^{2} \int_{-2}^{2} \cos x \cosh y \, dx \, dy = \sin 2 \sinh 2$$

$$\doteq 0.82447\ 37090\ 77809\ 15433$$

with $E = 0.0^{13}9401323$ and $C = 0.0^{13}9406250$.

These formulae clearly have high precision, even with considerable values of h.

10. Quadrature over a Cube; specially chosen points. The search for such formulae is more difficult in 3 or more dimensions. It seems that one or more extra available constants are needed in order to obtain real points. We shall not pursue this, but give one simple formula for three dimensions.

We find nothing convenient by use of points $\alpha(a)$, with or without 0; likewise 0 with $\beta(b)$ fails to give real points. We can, however, use 12 points $\beta(b)$ alone. We have then to satisfy

$$(10.1) \quad \begin{cases} 2n(n-1)A_\beta = 1 \\ 8b^4(4-n)A_\beta = \frac{4}{15}, \text{ where } n = 3. \end{cases}$$

This yields $b = (2/5)^{1/4} = 0.79527\ 07287\ 67051 \quad A_\beta = 1/12$

with main correction term

$$C = (156\, b^6\, A_\beta + \tfrac{16}{21})\ \frac{h^6}{6!}\ \mathfrak{J}^6\ f_0 = 0.00562\ h^6\ \mathfrak{J}^6\ f_0$$

With the example of §6, with integrand $\cos \frac{3}{4} x \cos y \cosh z$ (10.1) gives $J \doteq 0.97519$ with $E = -0.00489$ and $C = +0.00494$.

The only formula found that allows for the term $\mathcal{J}^6 f_0$ and has an error of order h^8 is (6.33), which needs 9 points. It is evident that further search is needed.

This work was supported in part by the Office of Naval Research under Contract Nonr-1834(27).